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# The topological string associated with a simple singularity of type $D$

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## ABSTRACT

The partition function of  $D_{N+1}$  topological string, the coupled system of topological gravity and  $D_{N+1}$  topological minimal matter , is proposed in the framework of KP hierarchy. It is specified by the elements of  $GL(\infty)$  which constitute the deformed family from the  $A_{2N-1}$  topological string. Its dispersionless limit is investigated from the view of both dispersionless KP hierarchy and singularity theory. In particular the free energy restricted on the small phase space coincides with that for the topological Landau-Ginzburg model of type  $D_{N+1}$ .

Recently much attention has been paid to the understanding of topological string theory. In particular the topological string of type  $A$  [1] has been studied from several perspectives. Its partition function  $e^{\mathcal{F}_A}$  is defined as the  $\tau$  function of Kadomtsev-Petriashvili (KP) hierarchy [2, 3] which satisfies [4] the genus expansion of the form :  $\mathcal{F}_A = \sum_{g=0}^{\infty} \mathcal{F}_{A,g}$ .  $\mathcal{F}_{A,g}$  is the free energy on the 2-surface of genus  $g$ . By taking an appropriate “small phase space”  $T_s$  of the space of KP times, the quantity  $\mathcal{F}_A|_{T_s}$  is shown [4, 5] to be deeply related with the versal deformation of a simple singularity of type  $A$ .

In spite of this progress of our understanding on the topological string of type  $A$ , the other noncritical topological strings associated with simple singularities have not been known well. In this letter we try to describe the partition function of the topological string which is associated with a simple singularity of type  $D$ . We begin in the section 1 by studying the deformation of topological string of type  $A$ . The dispersionless limit of this deformation is investigated in the section 2, which is shown to be deeply related with the versal deformation of a simple singularity of type  $D$ . In the last section we give some discussion and propose the partition function of topological string of type  $D$ .

## 1 Deformation of $A_{2N-1}$ topological string

The  $A_{2N-1}$  topological string is the coupled system of topological gravity and  $A_{2N-1}$  topological minimal matter [1]. Since the partition function  $e^{\mathcal{F}_{A_{2N-1}}}$ , which has been studied from the view of matrix integrals, is a  $\tau$  function of KP hierarchy [2], it can be realized in terms of free fermion system,  $\psi(z) = \sum_l \psi_l z^{-l}$ ,  $\psi^*(z) = \sum_l \psi_l^* z^{-l-1}$  [6]<sup>1</sup>:

$$\exp \left\{ \mathcal{F}_{A_{2N-1}}(t; \hbar) \right\} = \langle 0 | \exp \left\{ \sum_{k \not\equiv 0 \bmod 2N, k \geq 1} \frac{t_k}{\hbar} J_k \right\} g_0 | 0 \rangle, \quad (1)$$

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<sup>1</sup> $\{\psi_l, \psi_m^*\} = \delta_{l+m,0}$ .

where the KP time  $t_{2Nm+n}$  ( $m \geq 0, 1 \leq n \leq 2N-1$ ) is the parameter coupled with  $\sigma_m(\mathcal{O}_n)$ , the m-th gravitational descendants of primary field  $\mathcal{O}_n$  ( $\mathcal{O}_1 = P$  : puncture operator) and " $\hbar$ " plays the role of cosmological constant of this string theory.  $J(z) = \sum_l J_l z^{-l-1}$  is a  $U(1)$  current of this fermion system, that is,  $J(z) =: \psi\psi^* : (z)$ . The vacuum  $|0\rangle$  is introduced by the conditions :  $\psi_k |0\rangle = 0$  for  $k \geq 1$ , and  $\psi_k^* |0\rangle = 0$  for  $k \geq 0$ .  $g_0$  in (1) is an element of  $GL(\infty)$  which characterizes the partition function.

With the above brief review of the  $A_{2N-1}$  topological string let us study the following perturbation of this system :

$$\begin{aligned} \exp \left\{ \mathcal{F}(t_{odd}; t^{*2}; \hbar) \right\} &= \langle 0 | \exp \left\{ \sum_{k \geq 1} \frac{t_{2k-1}}{\hbar} J_{2k-1} \right\} g(t^{*2}) | 0 \rangle, \\ g(t^{*2}) &= g_0 \exp \left\{ \frac{t^{*2}}{2\hbar} J_{-1} \right\}, \end{aligned} \quad (2)$$

where  $t^*$  is a deformation parameter ( $g(0) = g_0$ ) and we set the even KP times equal to zero, that is,  $t_{2k} \equiv 0$  for  $\forall k \geq 1$ .  $t_{odd} = (t_1, t_3, \dots)$ . Notice that  $\mathcal{F}(t_{odd}; t^{*2}; \hbar)$  reduces to  $\mathcal{F}_{A_{2N-1}}(t_{odd}; \hbar)$  at  $t^* = 0$ . Let us describe  $g(t^{*2})$  in (2), an element of  $GL(\infty)$ , in some detail. For this purpose we shall introduce an infinite dimensional vector space  $\mathcal{V} = \bigoplus_{n \in \mathbf{Z}} \mathbf{C} e_{-n}(x)$  on which  $\hbar \partial_x$  and  $x$  act as <sup>2</sup>

$$\hbar \partial_x e_n(x) = e_{n-1}(x), \quad x e_n(x) = \hbar(n+1) e_{n+1}(x). \quad (3)$$

Notice that the fermion modes  $\psi_n, \psi_n^*$  and the vacuum  $|0\rangle$  can be written in terms of these bases  $\{e_{-n}(x)\}_{n \in \mathbf{Z}}$  :

$$\begin{aligned} |0\rangle &= e_{-1}(x) \wedge e_{-2}(x) \wedge \dots, \\ \psi_n &= e_{-n}(x) \wedge, \quad \psi_n^* = i_{e_n(x)}. \end{aligned} \quad (4)$$

With these correspondences any pseudo-differential operator can be realized by the free fermion. Say, for an example,  $J_k =: (\hbar \partial_x)^k :$  holds.  $\exp \left\{ \frac{t^{*2}}{2\hbar} J_{-1} \right\} \equiv$

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<sup>2</sup> $e_n(x)$  has the form:  $e_n(x) = \int d\lambda e^{\frac{x\lambda}{\hbar}} \lambda^{-n-1}$ .

$\exp \left\{ \frac{t^{*2}}{2\hbar} : (\hbar \partial_x)^{-1} : \right\}$  transforms  $e_{-n}(x)$  into

$$e_{-n}(x) \rightarrow \sum_{l \geq 0} \left( \frac{t^{*2}}{2\hbar} \right)^l \int d\lambda e^{\frac{x\lambda}{\hbar}} \lambda^{n-1-l}, \quad (5)$$

and the action of  $g_0$  on  $\mathcal{V}$  is [3]

$$e_{-n}(x) \rightarrow \phi_n(x) = \int d\lambda e^{\frac{x\lambda}{\hbar}} \hat{\phi}_n(\lambda), \quad (6)$$

where  $\hat{\phi}_n$  is given by

$$\hat{\phi}_n(\lambda) = \left( \frac{\lambda^{2N-1}}{\hbar} \right)^{\frac{1}{2}} e^{-\frac{2N}{2N+1} \frac{\lambda^{2N+1}}{\hbar}} \cdot \int dz z^{n-1} e^{\frac{1}{\hbar} \left\{ -\frac{z^{2N+1}}{2N+1} + \lambda^{2N} z \right\}}. \quad (7)$$

The integration with respect to  $z$  in (7) is performed by the saddle point method<sup>3</sup>. By combining these two transforms of  $e_{-n}(x)$  the action of  $g(t^{*2})$  on  $\mathcal{V}$  is described as

$$e_{-n}(x) \rightarrow \phi_n(x; t^{*2}) = \sum_{l \geq 0} \frac{1}{l!} \left( \frac{t^{*2}}{2\hbar} \right)^l \phi_{n-l}(x). \quad (8)$$

$\phi_n(x; t^{*2})$  has the following equivalent expression :

$$\begin{aligned} \phi_n(x; t^{*2}) &= \int d\lambda e^{\frac{x\lambda}{\hbar}} \hat{\phi}_n(\lambda; t^{*2}), \\ \hat{\phi}_n(\lambda; t^{*2}) &= \left( \frac{\lambda^{2N-1}}{\hbar} \right)^{\frac{1}{2}} e^{-\frac{2N}{2N+1} \frac{\lambda^{2N+1}}{\hbar}} \cdot \int dz z^{n-1} e^{\frac{1}{\hbar} \left\{ -\frac{z^{2N+1}}{2N+1} + \lambda^{2N} z + \frac{t^{*2}}{2} z^{-1} \right\}}, \end{aligned} \quad (9)$$

where the integration with respect to  $z$  is performed after the change of the variable :  $z \rightarrow z' = z - \lambda$ .

Because, for an each fixed value of  $t^*$ ,  $e^{\mathcal{F}(t_{odd}; t^{*2}; \hbar)}$  (2) can be regarded as a  $\tau$ -function of the KP hierarchy with even KP times setting zero ( $t_{2k} \equiv 0$ ), one can introduce  $W(t^{*2}; \hbar \partial_x)$ , the corresponding wave operator of KP hierarchy, as

$$W(t^{*2}; \hbar \partial_x) = 1 + \sum_{l \geq 1} w_l(t^{*2}; \hbar) (\hbar \partial_x)^{-l}, \quad (10)$$

where  $w_l(t^{*2}; \hbar) \equiv w_l(t_{odd}; t^{*2}; \hbar)$  and we identify  $t_1 = x$ . Through the element  $g(t^{*2}) \in GL(\infty)$  one can also associate  $e^{\mathcal{F}(t_{odd}; t^{*2}; \hbar)}$  (2) with a point of the

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<sup>3</sup>By the saddle point method  $\hat{\phi}_n(\lambda)$  has the form:  $\hat{\phi}_n(\lambda) = \sum_{l \geq 0} d_l^{(n)} \lambda^{n-l(2N+1)}$

Universal Grassmann manifold (UGM) [7].  $V(t^{*2})$ , the corresponding point of UGM, is given by

$$V(t^{*2}) = \oplus_{n \geq 1} \mathbf{C} \phi_n(x; t^{*2}). \quad (11)$$

The relation between  $V(t^{*2})$ , the underlying point of UGM, and  $W(t^{*2}; \hbar \partial_x)$ , the wave operator of KP hierarchy, is [8]

$$V(t^{*2}) = W_0(t^{*2}; \hbar \partial_x)^{-1} V_\emptyset, \quad (12)$$

where  $V_\emptyset$  is the point of UGM which corresponds to the vacuum  $|0\rangle$ , that is,  $V_\emptyset = \oplus_{n \geq 1} \mathbf{C} e_{-n}(x)$ , and  $W_0(t^{*2}; \hbar \partial_x) = W(t^{*2}; \hbar \partial_x)|_{t_{odd}=(x, 0, \dots)}$  is the initial value of the wave operator. We shall characterize the wave operator (10) from the study of the point of UGM (11). In particular we should notice that the bases  $\{\phi_n(x; t^{*2})\}_{n \in \mathbf{Z}}$  (8) satisfy the following set of equations:

$$F \phi_n(x; t^{*2}) = \phi_{n+2N}(x; t^{*2}) + \hbar(1-n)\phi_{n-1}(x; t^{*2}) + \frac{t^{*2}}{2}\phi_{n-2}(x; t^{*2}), \quad (13)$$

$$G \phi_n(x; t^{*2}) = -\phi_{n+1}(x; t^{*2}), \quad (14)$$

where we introduce the operators  $F$  and  $G$  as

$$F = (\hbar \partial_x)^{2N}, \quad G = \frac{1}{2N} \left\{ x - 2N(\hbar \partial_x)^{2N} - \frac{2N-1}{2} \partial_x^{-1} \right\} (\hbar \partial_x)^{-2N+1}. \quad (15)$$

By combining these equations with (12) we can see

$$P_0 = (\hbar \partial_x)^{2N} + x + \frac{t^{*2}}{2} (\hbar \partial_x)^{-2}, \quad (16)$$

$$Q_0 = -\hbar \partial_x, \quad (17)$$

where we set  $P_0 = W_0(t^{*2}; \hbar \partial_x) F W_0(t^{*2}; \hbar \partial_x)^{-1}$  and  $Q_0 = W_0(t^{*2}; \hbar \partial_x) G W_0(t^{*2}; \hbar \partial_x)^{-1}$ .

These two operators can be also regarded as  $P_0 = P|_{t_{odd}=(x, 0, \dots)}$  and  $Q_0 = Q|_{t_{odd}=(x, 0, \dots)}$ , the initial values of the operators  $P$  and  $Q$  under the flows of odd KP times :

$$P = L^{2N}, \quad (18)$$

$$Q = \frac{1}{2N} \left( M - 2NL^{2N} - \frac{2N-1}{2} \hbar L^{-1} \right) L^{-2N+1}. \quad (19)$$

In the above descriptions we introduce the Lax and Orlov operators of KP hierarchy [9] as

$$\begin{aligned} L &= W(t^{*2}; \hbar\partial_x) \hbar\partial_x W(t^{*2}; \hbar\partial_x)^{-1}, \\ M &= W(t^{*2}; \hbar\partial_x) \sum_{l \geq 1} (2l-1) t_{2l-1} (\hbar\partial_x)^{2l-2} W(t^{*2}; \hbar\partial_x)^{-1}. \end{aligned} \quad (20)$$

Since  $Q_0$  (17) is a differential operator and the time evolutions of  $Q$  (19) follow from those of the KP hierarchy <sup>4</sup>,

$$\hbar\partial_{t_{2k-1}} Q = [B_{2k-1}, Q] \quad \left( B_{2k-1} = (L^{2k-1})_+ \right), \quad (21)$$

this property of  $Q_0$  is preserved under the flows :

$$(Q)_- = 0. \quad (22)$$

At this stage it may be convenient to give some remarks. Let us attach the topological weight, "wt", to the parameters [4]:

$$wt(t_k) = k - 2N - 1, \quad wt(\hbar) = -2N - 1. \quad (23)$$

Notice that these topological weights reflect the ghost numbers of the observables for  $A_{2N-1}$  topological string [1]. In particular the action of  $g_0$  (6) on  $\mathcal{V}$  preserves these weights <sup>5</sup>. We shall also introduce the topological weight of  $t^*$  as

$$wt(t^*) = -N - 1, \quad (24)$$

so that the action of  $g(t^{*2})$  (8) preserves them. Consequently, from eqs. (16) and (17), the operators  $P$  and  $Q$  are quasi-homogeneous with respect to these topological weights (23) and (24):

$$wt(P) = -2N, \quad wt(Q) = -1, \quad (25)$$

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<sup>4</sup> ( )<sub>±</sub> are the projection operators with respect to  $\hbar\partial_x$  :

$$((\hbar\partial_x)^m)_+ = \begin{cases} (\hbar\partial_x)^m & m \geq 0 \\ 0 & m < 0 \end{cases} \quad ((\hbar\partial_x)^m)_- = (\hbar\partial_x)^m - ((\hbar\partial_x)^m)_+.$$

<sup>5</sup>One can also attach the topological weights to  $\lambda$  and  $z$  in (7) as  $wt(\lambda) = wt(z) = -1$ .

which lead to the quasi-homogeneity of  $L$  and  $M$  :

$$wt(L) = -1, \quad wt(M) = -2N. \quad (26)$$

Since one can expand the Orlov operator  $M$  in terms of the Lax operator  $L$  as <sup>6</sup>

$$M = \sum_{l \geq 1} (2l-1)t_{2l-1}L^{2l-2} - \hbar \sum_{l \geq 1} lP_l(-\hbar \tilde{\partial}_{t_{odd}})\mathcal{F}(t_{odd}; t^{*2}; \hbar)L^{-l-1}, \quad (27)$$

$\mathcal{F}(t_{odd}; t^{*2}; \hbar)$  is also quasi-homogeneous with respect to these topological weights:

$$wt(\mathcal{F}) = 0. \quad (28)$$

## 2 $\hbar \rightarrow 0$ limit

Let us study the  $\hbar \rightarrow 0$  limit of  $\mathcal{F}(t_{odd}; t^{*2}; \hbar)$  (2). The underlying integrable hierarchy is the dispersionless KP hierarchy [10],[11] which is a "quasi-classical" limit of the KP hierarchy in the following substitution :

$$[\hbar \partial_x, x] = \hbar \rightarrow \{p, x\} = 1, \quad (29)$$

where the Poisson bracket is defined by

$$\{f, g\} = \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial p}. \quad (30)$$

The  $\hbar \rightarrow 0$  limit of the Lax operator  $L$  and the Orlov operator  $M$  (20) have the form :

$$\begin{aligned} L \rightarrow \mathcal{L} &= p + \sum_{l \geq 1} u_{l+1}(t_{odd}; t^{*2})p^{-l} \\ M \rightarrow \mathcal{M} &= \sum_{l \geq 1} (2l-1)t_{2l-1}\mathcal{L}^{2l-2} + \sum_{l \geq 1} v_{2l}(t_{odd}; t^{*2})\mathcal{L}^{-2l}, \end{aligned} \quad (31)$$

which satisfy  $\{\mathcal{L}, \mathcal{M}\} = 1$ . Notice that  $v_{2l}$  in (31) is given by

$$v_{2l} = \frac{\partial \mathcal{F}_0}{\partial t_{2l-1}}, \quad (32)$$

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<sup>6</sup>  $P_l(t_{odd})$  is a Schur polynomial with  $t_{even} = 0$  which generating function is  $e^{\sum_{l \geq 1} t_{2l-1} \lambda^{2l-1}} = \sum_{l \geq 0} P_l(t_{odd}) \lambda^l$ , and  $\tilde{\partial}_{t_{odd}} = (\partial_{t_1}, \frac{\partial_{t_3}}{3}, \dots)$ .

where  $\mathcal{F}_0$  is the leading coefficient of the expansion of  $\mathcal{F}$  (2) in terms of  $\hbar$  [11]:

$$\mathcal{F}(t_{odd}; t^{*2}; \hbar) = \hbar^{-2} \left\{ \mathcal{F}_0(t_{odd}; t^{*2}) + o(\hbar) \right\}. \quad (33)$$

One can also say that  $e^{\mathcal{F}_0}$  is a  $\tau$  function of the dispersionless KP hierarchy.

We shall begin by studying the  $\hbar \rightarrow 0$  limit of the operator  $P$  (18). Obviously  $\mathcal{P}$ , the dispersionless limit of  $P$ , is given by  $\mathcal{P} = \mathcal{L}^{2N}$ . The initial value of  $\mathcal{P}$  is the dispersionless limit of the eq. (16) :

$$\mathcal{P}|_{t_{odd}=(x,0,\dots)} = p^{2N} + x + \frac{t^{*2}}{2} p^{-2}. \quad (34)$$

The time evolutions of  $\mathcal{P}$  follow from those of the dispersionless KP hierarchy<sup>7</sup> :

$$\frac{\partial \mathcal{P}}{\partial t_{2l-1}} = \{\mathcal{B}_{2l-1}, \mathcal{P}\}, \quad (\mathcal{B}_{2l-1} = (\mathcal{L}^{2l-1})_+) \quad (35)$$

Notice that  $\mathcal{P}$  is quasi-homogeneous with respect to the topological weights ( $wt(\mathcal{P}) = -2N$ ). Then, by considering these time evolutions with the initial value (34), we can conclude that  $\mathcal{P}$  has the following form :

$$\mathcal{P} = p^{2N} + \sum_{l=1}^N a_l(t_{odd}; t^{*2}) p^{2N-2l} + a_*(t_{odd}; t^{*2}) p^{-2}. \quad (36)$$

Nextly we shall look at the dispersionless limit of the operator  $Q$  (19). It becomes

$$\mathcal{Q} = \frac{1}{2N} (\mathcal{M} - 2N\mathcal{L}^{2N}) \mathcal{L}^{-2N+1}. \quad (37)$$

In particular the  $\hbar \rightarrow 0$  limit of the condition (22) is

$$(\mathcal{Q})_- = 0, \quad (38)$$

which is equivalent to

$$\text{res}\{(\mathcal{Q})_- \mathcal{L}^k \frac{\partial \mathcal{L}}{\partial p} dp\} = 0 \quad \text{for } \forall k \geq 0. \quad (39)$$

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<sup>7</sup> At the dispersionless limit  $( )_\pm$  become the projection operators with respect to  $p$  :

$$(p^m)_+ = \begin{cases} p^m & m \geq 0 \\ 0 & m < 0 \end{cases} \quad (p^m)_- = p^m - (p^m)_+.$$

One can rephrase the condition (39) into that on  $\mathcal{F}_0$ . By substituting the expansion of  $\mathcal{M}$  (31) with respect to  $\mathcal{L}$  into  $\mathcal{Q}$  (37) and then utilizing the formula [11]

$$\text{res}\{\mathcal{L}^{2m+1} \frac{\partial \mathcal{B}_{2m+1}}{\partial p} dp\} = \frac{\partial^2 \mathcal{F}_0}{\partial t_{2m+1} \partial t_{2n+1}}, \quad (40)$$

the condition (39) turns to

$$\frac{\partial C_0(t_{odd}; t^{*2})}{\partial t_{2k+1}} = 0 \quad \text{for } \forall k \geq 0, \quad (41)$$

where

$$\begin{aligned} C_0(t_{odd}; t^{*2}) &= \sum_{l \geq 1} (2l - 1 + 2N) t_{2l-1+2N} \frac{\partial \mathcal{F}_0}{\partial t_{2l-1}} - 2N \frac{\partial \mathcal{F}_0}{\partial t_1} \\ &\quad + \frac{1}{2} \sum_{l=1}^N (2l - 1)(2N - 2l + 1) t_{2l-1} t_{2N-2l+1}. \end{aligned} \quad (42)$$

Notice that one can integrate the eq.(41) to the form,  $C_0(t_{odd}; t^{*2}) + f(t^{*2}) = 0$ , that is, up to the ambiguity for the dependence of parameter  $t^*$ . From the quasi-homogeneity of  $C_0$  with respect to the topological weights this dependence will be restricted to

$$C_0(t_{odd}; t^{*2}) + \alpha t^{*2} = 0, \quad (43)$$

where "α" is a constant which still remains undetermined. We shall determine the value of α. Notice that, by cosidering the eq. (41) for the case of  $k = N$  and then restricting it on the initial time, we can obtain

$$\begin{aligned} \frac{\partial \mathcal{F}_0}{\partial t_1} \Big|_{t_{odd}=(x,0,\dots)} &= \frac{2N}{2N+1} \text{res}\{\mathcal{L}^{2N+1} dp\} \Big|_{t_{odd}=(x,0,\dots)} \\ &= \frac{1}{2} t^{*2}. \end{aligned} \quad (44)$$

The last equality in (44) is the result of the direct calculation based on the initial condition (34). One can also obtain the following relation by restricting the eq. (43) on the initial time :

$$\frac{\partial \mathcal{F}_0}{\partial t_1} \Big|_{t_{odd}=(x,0,\dots)} = \frac{\alpha}{2N} t^{*2}. \quad (45)$$

By comparing these two equations we obtain  $\alpha = N$ . Hence the condition  $(\mathcal{Q})_- = 0$  gives rise to the equation :

$$C_0(t_{odd}; t^{*2}) + Nt^{*2} = 0. \quad (46)$$

We shall introduce  $T_s^{KP}$ , the "small phase space" of odd KP times as  $T_s^{KP} \ni t_{odd} = (t_1, t_3, \dots) \Leftrightarrow t_{2N+2k+1} = 0 \text{ for } \forall k \geq 0$ . On this small phase space the following equalities hold : For  $1 \leq k \leq N$

$$t_{2N-2k+1} = \frac{2N}{(2N-2k+1)(2k-1)} \text{res}\{\mathcal{L}^{2k-1}dp\} \Big|_{T_s^{KP}}, \quad (47)$$

$$\frac{\partial \mathcal{F}_0}{\partial t_{2k-1}} \Big|_{T_s^{KP}} = \frac{2N}{2N+2k-1} \text{res}\{\mathcal{L}^{2N+2k-1}dp\} \Big|_{T_s^{KP}}. \quad (48)$$

Though these relations are the direct consequences of the equation (46), they are also deeply related with the singularity theory [12]. In order to explain this point, let us consider the versal deformation of a simple singularity of type  $D_{N+1}$  :

$$h(y, z) = y^N + yz^2 + \sum_{l=1}^N a_l y^{N-l} + bz, \quad (49)$$

where  $(a_1, a_2, \dots, a_N, b) \in \mathbf{C}^{\otimes N+1}$  are the deformation parameters. The flat coordinates  $s = (s_1, s_2, \dots, s_N, s^*)$  of this parameter space are given [13] by

$$s_k = \frac{2N}{2k-1} \sum_{\alpha_1, \dots, \alpha_N} \left( \begin{array}{c} \frac{2N-1}{N} \\ \sum_{l=1}^N \alpha_l \end{array} \right) \frac{(\sum_{l=1}^N \alpha_l)!}{\prod_{l=1}^N \alpha_l!} a_1^{\alpha_1} \cdots a_N^{\alpha_N}, \quad s^* = b, \quad (50)$$

where the sum is performed with respect to  $(\alpha_1, \dots, \alpha_N) \in (\mathbf{Z}_{\geq 0})^{\otimes N}$  which satisfies  $\sum_{l=1}^N l\alpha_l = k$ . Notice that the above expression for the flat coordinates can be simplified [5] as

$$s_k = \frac{2N}{2k-1} \text{res}\{H^{\frac{2k-1}{2N}}(p)dp\}, \quad \frac{s^{*2}}{2} = \text{res}\{pH(p)dp\}, \quad (51)$$

where we introduce the potential function  $H(p)$  as

$$H(p) = p^{2N} + \sum_{l=1}^N a_l p^{2N-2l} + \frac{b^2}{2} p^{-2}. \quad (52)$$

It is important to note that, with the identification of the potential  $H$  (52) with  $\mathcal{P}$  (36), the eqs. (47) and (51) tell us

$$s_k = (2N - 2k + 1)t_{2N-2k+1} \quad \text{on } T_s^{KP}. \quad (53)$$

What about the flat coordinate  $s^*$ ? One can write  $a_*(t_{odd}; t^{*2})$ , the coefficient of  $p^{-2}$  in  $\mathcal{P}$  (36), as <sup>8</sup>

$$\begin{aligned} a_*(t_{odd}; t^{*2}) &= \text{res}\{p\mathcal{L}^{2N}dp\} \\ &= \frac{2N}{2N+1} \text{res}\{\mathcal{L}^{2N+1}dp\} \\ &\quad - \sum_{l=1}^N \frac{1}{2l-1} \text{res}\{\mathcal{L}^{2l-1}dp\} \text{res}\{\mathcal{L}^{2N-2l+1}dp\}, \end{aligned} \quad (54)$$

from which, by using the eqs.(46), (47) and (48), one can obtain

$$\text{res}\{p\mathcal{L}^{2N}dp\} \Big|_{T_s^{KP}} = \frac{1}{2}t^{*2}. \quad (55)$$

Hence on the small phase space  $T_s^{KP} s = (s_1, \dots, s_N, s^*)$ , the flat coordinates of the versal deformation of  $D_{N+1}$  simple singularity, coincide with the KP times and the deformation parameter,  $(t_1, \dots, t_{2N-1}, t^*)$ .

### 3 Discussion

The eqs. (46), (47) and (48) mean that  $\mathcal{F}_0(t_{odd}; t^{*2}) \Big|_{T_s^{KP}}$  is equal to the free energy of  $D_{N+1}$  topological Landau-Ginzburg (LG) model [5]. The physical observables of  $D_{N+1}$  topological LG model are denoted as  $\tilde{\mathcal{O}}_k$  ( $1 \leq k \leq N$ ) and  $\mathcal{O}_*$ . The values of  $N=2$   $U(1)$  charge are  $\frac{k-1}{N}$  and  $\frac{N-1}{2N}$  respectively. The KP times  $t_{2k-1}$  ( $1 \leq k \leq N$ ) and the deformation parameter  $t^*$  are those coupled with these observables. Moreover one can see that the following relation is consistent with the condition (46) :

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$$\partial_{t_{2m-1}} \partial_{t_{2i-1}} \partial_{t_{2j-1}} \mathcal{F}_0$$

<sup>8</sup> We use the formula [11] :  $p = \mathcal{L} - \sum_{l \geq 1} \frac{1}{l} \text{res}\{\mathcal{L}^l dp\} \mathcal{L}^{-l}$ .

$$\begin{aligned}
&= \sum_{k=1}^N \frac{2m-1}{(2k-1)(2N-2k+1)} \partial_{t_{2(m-N)-1}} \partial_{t_{2k-1}} \mathcal{F}_0 \partial_{t_{2N-2k+1}} \partial_{t_{2i-1}} \partial_{t_{2j-1}} \mathcal{F}_0 \\
&+ \frac{2m-1}{N} \partial_{t_{2(m-N)-1}} \partial_{t^*} \mathcal{F}_0 \partial_{t^*} \partial_{t_{2i-1}} \partial_{t_{2j-1}} \mathcal{F}_0,
\end{aligned} \tag{56}$$

where  $m \geq N+1$  and  $1 \leq i, j \leq N$ . This can be regarded as the recursion relation on 2-sphere for the coupled system of topological matter and topological gravity [14]. Thus we can also say that the KP time  $t_{2Nm+2k-1}$  ( $m \geq 0, 1 \leq k \leq N$ ) is the parameter coupled with  $\sigma_m(\tilde{\mathcal{O}}_k)$ , the  $m$ -th gravitational descendant of  $\tilde{\mathcal{O}}_k$ . With these observations it is very plausible that our model gives the description of  $D_{N+1}$  topological string. So we would like to propose the following conjecture.

**Conjecture 1**  $\mathcal{F}$  in (2) satisfies

$$\mathcal{F}(t_{odd}; t^{*2}; \hbar) = \sum_{g=0}^{\infty} \hbar^{2g-2} \mathcal{F}_g(t_{odd}; t^{*2}). \tag{57}$$

where  $\mathcal{F}_g$ , the free energy of  $D_{N+1}$  topological string on the 2-surface of genus  $g$ , has the form :

$$\begin{aligned}
&\mathcal{F}_g(t_{odd}; t^{*2}) \\
&= \sum_{s, d_*; \{l_i, k_i, d_i\}} \langle \mathcal{O}_*^{d_*} \prod_{i=1}^s \sigma_{l_i} (\tilde{\mathcal{O}}_{k_i})^{d_i} \rangle_{\overline{\mathcal{M}}_{g, d_* + \sum_{i=1}^s d_i}} \frac{(t^*)^{d_*} \prod_{i=1}^s (t_{2Nl_i+2k_i-1})^{d_i}}{d_*! \prod_{i=1}^s d_i!}.
\end{aligned} \tag{58}$$

$\langle \cdots \rangle_{\overline{\mathcal{M}}_{g, d_* + \sum_{i=1}^s d_i}}$  is the topological correlation function [15] evaluated on the compactified moduli space  $\overline{\mathcal{M}}_{g, d_* + \sum_{i=1}^s d_i}$  of the Riemann surface  $\Sigma$  of genus  $g$  with  $d_* + \sum_{i=1}^s d_i$  punctures. Notice that, because of the quasi-homogeneity of  $\mathcal{F}$  (28), one can see that the correlation function,  $\langle \mathcal{O}_*^{d_*} \prod_{i=1}^s \sigma_{l_i} (\tilde{\mathcal{O}}_{k_i})^{d_i} \rangle_{\overline{\mathcal{M}}_{g, d_* + \sum_{i=1}^s d_i}}$ , automatically vanishes except for the case :

$$\sum_{i=1}^s d_i \left( l_i + \frac{k_i - 1}{N} - 1 \right) + d_* \left( \frac{N-1}{2N} - 1 \right) = (d_{D_{N+1}} - 3)(1-g), \tag{59}$$

where  $d_{D_{N+1}} = \frac{N-1}{N}$ . This is nothing but the ghost number conservation rule for the coupled system of  $D_{N+1}$  topological minimal matter and topological gravity.

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